International Journal of Theoretical Physics, Vol. 39, No. 9, 2000

Four Quantum Conservation Laws for Black Hole Stationary Equilibrium Radiation Processes

S. Q. Wu[1] and X. Cai[1]

Received January 22, 2000

The classical first law of thermodynamics for a Kerr–Newman black hole (KNBH) is generalized to a law in quantum form on the event horizon. Then four quantum conservation laws on the KNBH equilibrium radiation process are derived. The Bekenstein–Hawking relation $\mathcal{S} = \mathcal{A}/4$ is exactly established. It can be inferred that the classical entropy of black hole arises from the quantum entropy of field quanta or quasiparticles inside the hole.

It has been a quarter century since Bekenstein [1] and Hawking [2] first showed that the entropy of a black hole is one fourth of its surface area. Despite considerable effort [3] on the quantum [4], dynamic [5], and statistical [6] origins of black hole thermodynamics, the exact source and mechanism of the Bekenstein–Hawking black hole entropy remain unclear [7].

By using the brick wall model, G 't Hooft [8] identified the black hole entropy with the entropy of a thermal gas of quantum field excitations outside the event horizon, whereas Frolov and Novikov [4] argued that the black hole entropy can be obtained by identifying the dynamical degrees of freedom of a black hole with the states of all fields which are located inside the black hole. A black hole acts as classical thermodynamic object, but its true microscopic structure is unknown [9].

Here we first derive the thermal spectrum and microscopic entropy of a massive complex scalar field on a Kerr–Newman black hole (KNBH) background. From this quantum entropy, we propose a quantum first law of black hole thermodynamics. Then we consider a system in which a KNBH is in equilibrium with this scalar field. Using a thermodynamic method, we

[1] Institute of Particle Physics, Hua-Zhong Normal University, Wuhan, 430079, China; e-mail: sqwu@iopp.ccnu.edu.cn,xcai@wuhan.cngb.com

0020-7748/00/0900-2215$18.00/0

obtain four conservation laws on black hole thermal radiation equilibrium process for energy, charge, angular momentum, and entropy, respectively. The total quantities of these quantum numbers of the whole system are conserved in the stationary thermal equilibrium radiation process. By identifying the complex scalar field with quasiparticles excited by the hole, we propose that the classical entropy of a black hole originates microscopically from the entropy of quanta which constitute the hole.

The general stationary axisymmetric solution to the Einstein equation is a rotating charged black hole (KNBH) described by three parameters: mass M, charge Q, and specific angular momentum $a = J/M$. So we deal with a sourceless, charged, massive scalar field with mass μ and charge q on this background in the nonextreme case ($0 < \varepsilon = \sqrt{M^2 - a^2 - Q^2} \leq M$; we use Planck unit, $G = \hbar = c = k_B = 1$).

In Boyer–Lindquist coordinates, a complex scalar wave function Ψ has a solution in variables separable form [10]

$$\Psi(t, r, \theta, \varphi) = R(r)S(\theta)e^{i(m\varphi - \omega t)} \tag{1}$$

Here the angular wave function $S(\theta)$ is an ordinary spheroidal function with spin weight $s = 0$ which satisfies the Legendre wave equation [11]:

$$\frac{1}{\sin\theta}\,\partial_\theta[\sin\theta\;\partial_\theta S(\theta)] + \left[\lambda - \frac{m^2}{\sin^2\theta} - (ka)^2\sin^2\theta\right]S(\theta) = 0 \tag{2}$$

while the radial wave function $R(r)$ is a modified generalized spheroidal wave function with an imaginary spin weight which satisfies the following "modified" generalized spin-weighted spheroidal wave equation of imaginary number order [12, 13]:

$$\begin{aligned}&\partial_r[(r - r_+)(r - r_-)\partial_r R(r)] + [k^2(r - r_+)(r - r_-)\\&\quad + 2(A\omega - M\mu^2)(r - M) + \frac{[A(r - M) + \varepsilon B]^2}{(r - r_+)(r - r_-)}\\&\quad + (2\omega^2 - \mu^2)(2M^2 - Q^2) - 2qQM\omega - \lambda]R(r) = 0\end{aligned} \tag{3}$$

where λ is a separation constant, and $r_\pm = M \pm \varepsilon$, $k^2 = \omega^2 - \mu^2$, $A = 2M\omega - qQ$, and $\varepsilon B = \omega(2M^2 - Q^2) - qQM - ma$.

When considering the thermal radiation of a KNBH, we need the asymptotic solutions of the radial function $R(r)$ at its event horizon $r = r_+$. In fact, the radial equation has two solutions whose indices at its regular singularity $r = r_+$ are $\pm iW$, where W is given blew. These two asymptotic solutions are

$$R(r) \sim (r - r_+)^{\pm iW} \qquad \text{when} \quad r \to r_+ \tag{4}$$

According to the analytical continued method suggested by Damour and

Ruffini [14], these two solutions differ by a extra factor $e^{2\pi W}$. It is easy to obtain a thermal radiation spectrum [15, 16] on the event horizon $r = r_+$:

$$\langle N \rangle = \frac{1}{e^{4\pi W} - 1}, \qquad W = \frac{\omega - m\Omega - q\Phi}{2\kappa} \tag{5}$$

where the surface gravity is $\kappa = (r_+ - M)/\mathscr{A}$, the angular velocity is $\Omega = a/\mathscr{A}$, the electrical potential is $\Phi = Qr_+/\mathscr{A}$, and the reduced horizon area is $\mathscr{A} = r_+^2 + a^2$.

Equation (5) demonstrates that a KNBH has an exact thermal property characterized by a temperature $T = \kappa/(2\pi)$ as common blackbody radiation does. The radiation modes of a complex scalar field are characterized by a frequency ω, a charge q, and an azimuthal quantum number m. This scalar field is rotating with an azimuthal angular velocity Ω and has a chemical potential Φ. The emitted scalar quanta obey Bose–Einstein statistics. Here, we adopt Bellido's [15] proposition that quantum number W is the quantum entropy of scalar fields on the KNBH background. The quantum entropy satisfies Bekenstein's [1] first law of black hole thermodynamics:

$$\omega = 2\kappa W + m\Omega + q\Phi \tag{6}$$

and we call this law as the quantum first law of black hole thermodynamics in integral form.

When a KNBH is in thermal equilibrium with a complex scalar field at temperature $T = \kappa/(2\pi)$, we could regard the hole's surface gravity, angular velocity, and electrical potential as external parameters that remain fixed or at most undergo a minute change which can be neglected under our present consideration. This assumption gives the following conditions of thermodynamic stable equilibrium of the system on the event horizon:

$$\kappa_{r_+ + 0} = \kappa_{r_+ - 0}, \qquad \Omega_{r_+ + 0} = \Omega_{r_+ - 0}, \qquad \Phi_{r_+ + 0} = \Phi_{r_+ - 0}$$

In this thermodynamic equilibrium system, the hole still emits and absorbs quanta although its parameters remain fixed. However, this stationary process is a detailed balance process [17], that is, the number of quanta emitted by the hole is equal to that absorbed by it. So the hole could preserve its parameters unchanged. Relative to the fixed parameters of the hole, due to vacuum polarization, the scalar field has some minute fluctuations which can be given by differentiating the energy relation of Eq. (6). Thus, we have the quantum first law of quantum thermodynamics in differential form:

$$\Delta\omega = 2\kappa\, \Delta W + \Omega\, \Delta m + \Phi\, \Delta q \tag{7}$$

Combining Eq. (7) with the following classical first law of black hole thermodynamics in differential form [17],

$$\Delta M = \frac{\kappa}{2}\,\Delta\mathscr{A} + \Omega\,\Delta J + \Phi\,\Delta Q \tag{8}$$

one can deduce four quantum conservation laws for energy, angular momentum, charge, and entropy, respectively:

$$\Delta M = n\,\Delta\omega \qquad \text{(energy)} \tag{9}$$

$$\Delta J = n\,\Delta m \qquad \text{(angular momentum)} \tag{10}$$

$$\Delta Q = n\,\Delta q \qquad \text{(charge)} \tag{11}$$

$$\Delta\mathscr{A}/4 = n\,\Delta W \qquad \text{(entropy)} \tag{12}$$

Here n is a multiplier needed to be determined further.

Equations (9)–(12) indicate that a KNBH has discrete increments of energy, angular momentum, charge, and entropy. That is, when a black hole emits particles, its energy, charge, angular momentum, and entropy are carried away by these quanta, and vice versa. These microscopic laws are only the reformulated detailed balance principle on stationary equilibrium process of black hole radiation. In a stationary thermal equilibrium radiation process, it is reasonable physically to conceive that what the hole gains the radiation loses. Thus, the total quantities of energy, charge, angular momentum, and entropy of the whole system remain conserved in this thermodynamic process.

However, this thermal equilibrium is in general unstable due to the existence of statistical fluctuations [18]. A minute perturbation will result in the hole completely evaporating to scalar field quanta or radiation quanta being absorbed fully by the hole. In the former case, the energy, charge, angular momentum, and entropy in the whole system will convert to those of the scalar field, in the latter case to those of the hole. But these physical quantities should be equal in these two extreme cases. From this, we could infer that a black hole consists of some elementary quasiexcitations, although at present we do not know what they really are. In this paper, we relate them to the scalar field quanta. When considering all modes of field excitations, the above conservation relations (9)–(12) must include summation with respect to all possible modes of field quanta.

Further, combining Eq. (6) with integral Smarr formula [19]

$$M = \kappa\mathscr{A} + 2J\Omega + Q\Phi \tag{13}$$

we can obtain a special quantum state $nm = J$, $n\omega = M/2$, $nq = Q/2$, $nW = \mathscr{A}/4$. As quantum numbers, m, ω, q, W are discrete numbers, not only the parameters J, M, Q, $\mathscr{A}$, but also ΔJ, ΔM, ΔQ, $\Delta\mathscr{A}$ must take discrete values. This means that a quantum KNBH can be thought of microscopically as consisting of all possible quasiparticles inside the hole, which is identified by us with all possible modes of bosonic field quanta having energy 2ω, charge $2q$, angular momentum m, and entropy $4W$ as elementary units.

In fact, Eq. (12) is a generalized second thermodynamic law in quantum form. By integrating this equation, we obtain the quantum black hole entropy:

$$nW = \frac{1}{4}\mathcal{A} + C \tag{14}$$

As the Bekenstein–Hawking classical black hole entropy [1, 2] is $S = A/4 = \pi\mathcal{A}$, the quantum entropy nW is equal to the reduced entropy $nW = \mathcal{S} = S/(4\pi)$, so we have the Bekenstein–Hawking relation (choose constant $C = 0$):

$$\mathcal{S} = nW = \mathcal{A}/4 \tag{15}$$

Equation (15) shows that the Bekenstein–Hawking black hole entropy is equal to the quantum entropy of a complex scalar field. On these grounds, one can conjecture that the classical entropy of black holes originates statistically from the quantum entropy of quantized fields.

In summary, we have considered the thermodynamics of a system consisting of a complex scalar field in thermal equilibrium with a Kerr–Newman black hole. Using the thermodynamic equilibrium condition on the event horizon, we derive four quantum conservation laws for black hole equilibrium radiation processes. The total energy, total charge, total angular momentum, and total entropy of the whole system are conserved in this process. By identifying the interior structure of a KNBH with a collection of quasiparticles, we infer that the classical entropy of a black hole originates microscopically from the quantum entropy of quanta inside the hole. However, this is still an open question that needs to be clarified.

ACKNOWLEDGMENTS

This work was supported in part by the NSFC and Hubei-NSF, China.

REFERENCES

1. J. D. Bekenstein (1973). *Phys. Rev. D.* **7,** 2333; (1974) **9**, 3292.
2. S. W. Hawking (1975). *Commun. Math. Phys.* **43,** 199.
3. J. Bekenstein, Do we understand black hole entropy?, gr-qc/9409015.
4. L. Bombelli, R. Koul, J. Lee, and R. Sorkin (1986). *Phys. Rev. D* **34**, 373.
5. V. Frolov and I. Novikov (1993). *Phys. Rev. D* **48**, 4545, gr-qc/9309001; V. Frolov (1995). *Phys. Rev. Lett.* **74**, 3319, gr-qc/9406037.
6. W. H. Zurek and K. S. Thorne (1985). *Phys. Rev. Lett.* **54**, 2171.
7. M. H. Lee and J. Kim (1996). hep-th/9604130; *Phys. Lett. A* **212**, 323, hep-th/9602129; J. Ho, W. T. Kim, Y. J. Park, and H. Shin (1997). *Class. Quant. Grav.* **14**, 2617, gr-qc/9704032.

8. G. ’t Hooft (1985). *Nucl. Phys. B* **256**, 727; L. Susskind and J. Uglum (1994). *Phys. Rev. D* **50**, 2700, hep-th/9401070.
9. M. Maggiore (1994). *Nucl. Phys. B* **429**, 205, gr-qc/9401027.
10. B. Carter (1968). *Phys. Rev.* **174**, 1559.
11. P. Moon and D. E. Spencer, *Field Theory Handbook*, 2nd ed. (Springer-Verlag, New York, 1971); P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, (McGraw-Hill, New York, 1953); M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, 9th ed. (Dover, New York, 1972).
12. S. Q. Wu and X. Cai (1999). *J. Math. Phys.* **40**, 4538, gr-qc/9904037.
13. E. W. Leaver (1986). *J. Math. Phys.* **27**, 1238.
14. T. Damour and R. Ruffini (1976). *Phys. Rev. D* **14**, 332.
15. J. G. Bellido, hep-th/9302127.
16. G. W. Gibbons and M. J. Perry (1978). *Proc. R. Soc. Lond. A* **358**, 467.
17. D. Bekenstein and A. Meisels (1976). *Phys. Rev. D* **15**, 2775.
18. L. Liu, *General Relativity* (Advanced Education Press, Beijing, 1987).
19. L. Smarr (1973). *Phys. Rev. Lett.* **30**, 71; 521(E).